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Convergence of the Derivatives of Hermite–Fejér Interpolation Polynomials of Higher Order Based at the Zeros of Freud Polynomials

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We shall prove pointwise convergence of the derivatives of Hermite–Fejér interpolation polynomials of higher order based at the zeros of orthonormal polynomials with respect to Freud weights $\exp(-x^m)$, $m = 2, 4, 6, \ldots$. © 1995 Academic Press, Inc.

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The purpose of this paper is to prove pointwise convergence of the derivatives of Hermite–Fejér interpolation polynomials of higher order based at the zeros of orthonormal polynomials with respect to a Freud weight of the form $\exp(-x^m)$ with an even positive integer m.

Let

$$Q(x) = \tfrac{1}{2}x^m, \qquad w(x) = \exp(-Q(x)),$$

where $m = 2, 4, 6, \ldots$. The orthonormal polynomials $p_n(w^2; x) = p_n(x) = \gamma_n x^n + \cdots$, where $\gamma_n > 0$, are defined by the relation

$$\int_{-\infty}^{\infty} p_l(x)\, p_n(x)\, w^2(x)\, dx = \delta_{ln}.$$

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These polynomials were investigated by Freud, e.g., [2, 3], and recently by many authors in connection with approximation theory. For detailed references and an extensive survey, readers may refer to Nevai [11].

We denote the zeros of $p_n(x)$ by x_{kn}, $k = 1, 2, ..., n$, where

$$x_{1n} > w_{2n} > \cdots > x_{nn}.$$

Let ν be a positive integer, and let l be a non-negative integer such that $\nu - 1 \geqslant l$. For $f \in C^l(\mathbf{R})$, the Hermite–Fejér interpolation polynomial $L_n(l, \nu; f, x)$ of order (l, ν) based at the zeros $x_{1n}, ..., x_{nn}$ is defined to be the unique algebraic polynomial of degree at most $\nu n - 1$ which satisfies

$$\begin{aligned} L_n(l, \nu; f, x_{kn}) &= f(x_{kn}), \\ L_n'(l, \nu; f, x_{kn}) &= f'(x_{kn}), ..., \\ L_n^{(l)}(l, \nu; f, x_{kn}) &= f^{(l)}(x_{kn}), \\ L_n^{(l+1)}(l, \nu; f, x_{kn}) &= 0, ..., \\ L_n^{(\nu-1)}(l, \nu; f, x_{kn}) &= 0 \end{aligned}$$

for $k = 1, 2, ..., n$. It is known that, for every $n = 1, 2, ...,$ $k = 1, 2, ..., n$ and $r = 0, 1, ..., \nu - 1$, there exists a unique polynomial $h_{rkn}(\nu; x)$ of degree $\nu n - 1$ satisfying

$$h_{rkn}^{(j)}(\nu; x_{pn}) = \delta_{rj}\, \delta_{kp}, \qquad p = 1, 2, ..., n, \qquad j = 0, 1, ..., \nu - 1$$

(cf. [8, Chap. I, Sect. 4]). The interpolation polynomial $L_n(l, \nu; f, x)$ is written in the form

$$L_n(l, \nu; f, x) = \sum_{k=1}^{n} \sum_{r=0}^{l} f^{(r)}(x_{kn})\, h_{rkn}(\nu; x).$$

Since $L_n(l, \nu; f, x) = 1$ for $f(x) = 1$, we see that

$$\sum_{k=1}^{n} h_{0kn}(\nu; x) = 1.$$

We note that $L_n(0, 1; f, x)$ is the Lagrange interpolation polynomial based at the points $x_{1n}, ..., x_{nn}$. We define the modulus of continuity of $f \in C(\mathbf{R})$ on an interval $[a, b]$ by $\omega([a, b]; f; h) = \sup\{|f(x) - f(y)|;\ |x - y| \leqslant h,\ x, y \in [a, b]\}$, $h > 0$.

Freud [4] and Nevai [9, 10] considered pointwise convergence of the Lagrange interpolation polynomials $L_n(0, 1; f, x)$ for the Hermite weight $\exp(-x^2)$, i.e., $m = 2$. Knopfmacher [6] estimated the rate of approximation of pointwise convergence of the polynomials $L_n(0, 1; f, x)$ for the class of regular Freud weights which includes the weights $\exp(-x^m)$, $m=2, 4, 6, \ldots$. Recently, the authors [5] observed the behavior of pointwise convergence of Hermite–Fejér interpolation polynomials $L_n(0, v; f, x)$ of order $(0, v)$ for the weights $\exp(-x^m)$, $m=2, 4, 6, \ldots$, and showed that if v is even then for every continuous function $f(x)$, the sequence $\{L_n(0, v; f, x)\}$ converges uniformly to $f(x)$ on any compact interval, and showed that if v is odd then for every interval I, there exists a continuous function $f(x)$ such that $\limsup_{n\to\infty} \max_{x\in I} |L_n(0, v; f, x)| = \infty$. On the other hand, Balázs [1] treated convergence problems of the derivatives $L_n^{(j)}(0, 1; f, x)$ of Lagrange interpolation polynomials for $m=2$, and proved that $|f^{(j)}(x) - L_n^{(j)}(0, 1; f, x)| \leqslant C\omega(\mathbf{R}; f^{(r)}; n^{-1/2})\, n^{-r/2+j}\{\log n + \exp(x^2/2)\}$ for $|x| \leqslant x_{1n}$, $j=0, \ldots, r$. In this paper, we shall consider convergence problems of the derivatives $L_n^{(j)}(l, v; f, x)$ for arbitrary $v=1, 2, \ldots, 0 \leqslant l \leqslant v-1$ and $m=2, 4, 6, \ldots$.

Let q_n denote the unique positive solution of the equation $q_n Q'(q_n) = n$, that is,

$$q_n = \left(\frac{2n}{m}\right)^{1/m}.$$

Our theorem is as follows:

THEOREM. *Let v be a positive integer and let l be an integer such that $v-1 \geqslant l \geqslant 0$. Then, there exist positive constants c and K satisfying the following:*

(i) *The case $v-1=l$: Let N be an integer such that $N \geqslant l$, and $f \in C^N(\mathbf{R})$. Then, for $|x| \leqslant cq_n$,*

$$\begin{aligned}|L_n^{(j)}(v-1, v; f, x) - f^{(j)}(x)| &\leqslant C(1+|x|^{j(m-2)/2})\, e^{vx^m/2} \\ &\quad\times \omega\left([-Kq_n, Kq_n]; f^{(N)}; \frac{q_n}{n}\right) \\ &\quad\times \left(\frac{q_n}{n}\right)^N n^j \log n \\ &\qquad j=0, 1, \ldots, N \qquad n=N+1, N+2, \ldots.\end{aligned}$$

(ii) *The case $v-1>l$: Let $f \in C^l(\mathbf{R})$. Then, for $|x| \leqslant cq_n$,*

$$\begin{aligned}|L_n^{(j)}(l, v; f, x) - f^{(j)}(x)| &\leqslant C(1+|x|^{j(m-2)/2})\, e^{vx^m/2} \\ &\quad\times \omega\left([-Kq_n, Kq_n]; f^{(l)}; \frac{q_n}{n}\right) \\ &\quad\times \left(\frac{q_n}{n}\right)^l n^j \log n \\ &\qquad j=0, 1, \ldots, l, \qquad n=l+1, l+2, \ldots.\end{aligned}$$

Here, C is a positive constant independent of n, x and f.

COROLLARY. (i) *The case* $v-1=l$: *Let* $N \geqslant l$. *If* $\lim_{h\to 0}\omega(\mathbf{R}; f^{(N)}; h) \log h = 0$, *then for every* $M>0$,

$$\lim_{n\to\infty} \max_{|x|\leqslant M} |L_n^{(j)}(v-1, v; f, x) - f^{(j)}(x)| = 0$$

for $j=0, 1, ..., [(1-1/m)N]$ (*the integral part of* $(1-1/m)N$).

(ii) *The case* $v-1>l$: If $\lim_{h\to 0}\omega(\mathbf{R}; f^{(l)}; h) \log h = 0$, *then for every* $M>0$,

$$\lim_{n\to\infty} \max_{|x|\leqslant M} |L_n^{(j)}(l, v; f, x) - f^{(j)}(x)| = 0$$

for $j=0, 1, ..., [(1-1/m)l]$.

We remark that the condition $\lim_{h\to 0}\omega(\mathbf{R}; f^{(N)}; h) \log h = 0$ holds, e.g., if $f^{(N)} \in \operatorname{Lip}\alpha$, $0<\alpha\leqslant 1$. We mention that Balázs [1] has obtained the estimate mentioned above for $v=1$ and $m=2$.

For the proof of Theorem, we need a basic estimate given in the following:

PROPOSITION. *Let* $r=0, 1, ..., v-1$. *There exists a positive constant* κ *such that*

$$\sum_{k=1}^{n} |h_{rkn}(v; x)| \leqslant C e^{v x^m/2} \left(\frac{q_n}{n}\right)^r \log n \tag{1.1}$$

for $x \in [-\kappa q_n, \kappa q_n]$ *and* $n=1, 2, ...$, *where* C *is a constant independent of* x *and* n.

We remark that for $m=2$ and $v=1$ (and thus $r=0$), Freud [4, Theorem 1] has gotten the estimate $\sum_{k=1}^{n} |h_{0kn}(1; x)| \leqslant C\{\log n + \exp(x^2/2)\}$.

The proofs of Theorem and Proposition will be given in the next section. We summarize here some known results which are needed in the proofs.

(I) [6, Lemma 4.11]: (i) There exists a constant $K_1>0$ independent of n such that $x_{1n} \leqslant K_1 q_n$, $n=1, 2, ...$.

(ii) There exist constants $C_1, C_2, \kappa_1 > 0$ independent of n and k such that $C_1 q_n/n < x_{k-1n} - x_{kn} < C_2 q_n/n$ for $x_{k-1n}, x_{kn} \in [-\kappa_1 q_n, \kappa_1 q_n]$.

Let $x_{(x,n)}$ denote the closest zero of $p_n(x)$ to x. If x is the midpoint of two zeros, then we define $x_{(x,n)}$ to be the closest zero of $p_n(x)$ on the left.

(II) [6, Theorem 3.7]: There exist constants $C_3, C_4, \kappa_2 > 0$ independent of n and x such that

$$C_3\,|x - x_{(x,n)}|\,\frac{n}{q_n}\,q_n^{-1/2} \leqslant |p_n(x)|\,w(x) \leqslant C_4\,|x - x_{(x,n)}|\,\frac{n}{q_n}\,q_n^{-1/2},$$

$$n = 1, 2, \ldots \text{ for } x \text{ with } |x| \leqslant \kappa_2 q_n.$$

(III) Bernstein's inequality [12, 4.8(51)]: Let $\Delta_n(t) = n^{-1}(1-t^2)^{1/2} + n^{-2}$. Let $R_n(t)$ be a polynomial of degree n. Then, for $-1 \leqslant t \leqslant 1$ and $j = 0, 1, \ldots,$

$$|R_n^{(j)}(t)| \leqslant C_5 \Delta_n(t)^{-j} \max_{|s| \leqslant 1} |R_n(s)|, \qquad n = 1, 2, \ldots,$$

where C_5 is a positive constant depending only on j.

(IV) [7, Corollary 1, Theorem 3]: Let $r = 0, 1, \ldots,$ and $g(t) \in C^r(\mathbf{R})$. Let $R_n^*(t)$ be the polynomial of best approximation of order n to $g(t)$ on the interval $[-1, 1]$. Then, for $|t| \leqslant 1$,

(i) $$|g^{(j)}(t) - R_n^{*(j)}(t)| \leqslant C_6 n^{-r} \Delta_n(t)^{-j} E_{n-r}(g^{(r)}; [-1, 1]),$$
$$j = 0, 1, \ldots, r,\ n = r+1, r+2.$$

(ii) $$|R_n^{*(j)}(t)| \leqslant C_7 n^{-r} \Delta_n(t)^{-j}\, \omega\left([-1, 1]; g^{(r)}; \frac{1}{n}\right),$$
$$j = r+1, r+2, \ldots, n = 1, 2, \ldots,$$

where C_6 is a positive constant depending only on r and C_7 is a positive constant depending only on j, and $E_{n-r}(g^{(r)}; [-1, 1]) = \max_{|t| \leqslant 1} |g^{(r)}(t) - T_{n-r}^*(t)|$, where $T_{n-r}^*(t)$ is the polynomial of best approximation of degree $n-r$ to $g^{(r)}(t)$.

Throughout this paper, the letters $C_1 \sim C_6$, K_1, κ_1, κ_2 with subscript are always the constants in the properties (I) $\sim$ (IV). For the rest of the paper, the letter C denotes a positive constant which may differ at each different occurrence, even in the same chain of inequalities.

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Let $l_{kn}(x)$, $k = 1, 2, \ldots$ be the fundamental polynomial of Lagrange interpolation polynomial $L_n(0, 1; f, x)$, that is, $l_{kn}(x) = h_{0kn}(1; x)$. Then,

$$l_{kn}(x) = \frac{p_n(x)}{(x - x_{kn})\,p_n'(x_{kn})}, \qquad k = 1, 2, \ldots, n. \tag{2.1}$$

We note that $h_{rkn}(\nu; x)$ is divided by $l_{kn}^{\nu}(x)$ $(=\{l_{kn}(x)\}^{\nu})$ and $x = x_{kn}$ is a root with multiplicity r of $h_{rkn}(\nu; x)$. We define $e_{ir}(\nu; k, n)$, $i = r, r+1, ..., \nu-1$ to be the coefficients in the expression

$$h_{rkn}(\nu; x) = l_{kn}^{\nu}(x) \sum_{i=r}^{\nu-1} e_{ir}(\nu; k, n)(x - x_{kn})^i,$$

$$k = 1, 2, ..., n. \tag{2.2}$$

After this, if there is no possibility of misunderstanding, we write briefly

$$x_k = x_{kn}; \qquad L_n(x) = L_n(l, \nu; f, x); \qquad h_{rk}(x) = h_{rkn}(\nu; x);$$

$$l_k(x) = l_{kn}(x); \qquad e_{ir}(k) = e_{ir}(\nu; k, n); \qquad \omega(h) = \omega([a, b]; f; h).$$

We first prove the Proposition. By (2.1) and (2.2), we have

$$\sum_{k=1}^{n} |h_{rk}(x)| \leqslant \sum_{i=r}^{\nu-1} \sum_{k=1}^{n} \left| \frac{p_n(x)}{(x - x_k)\, p_n'(x_k)} \right|^{\nu} |e_{ir}(k)|\, |x - x_k|^i$$

$$:= \sum_{i=r}^{\nu-1} \sum_{k=1}^{n} R_k(i, r, n; x).$$

Our task is to estimate $\sum_{k=1}^{n} R_k(i, r, n; x)$. To do so, we shall divide the sum into three parts. Here, we need a lemma on the behavior of $p_n'(x)$ in a neighborhood of x_k.

LEMMA 1 [5, Lemma 1]. *There exist constants $\tilde{\delta} > 0$ and $\tilde{\kappa} > 0$ such that $k < n$, $x_k \in [-\tilde{\kappa} q_{n-1}, \tilde{\kappa} q_{n-1}]$ and $x_k - \tilde{\delta} q_n/n \leqslant x \leqslant x_k + \tilde{\delta} q_n/n$, then*

$$C \frac{n}{q_n} q_n^{-1/2} w(x_k)^{-1} \leqslant |p_n'(x)| \leqslant C \frac{n}{q_n} q_n^{-1/2} w(x_k)^{-1},$$

where C is independent of k, n and x.

By (I), we may suppose that the constant $\tilde{\kappa}$ satisfies $x_n < -\tilde{\kappa} q_n$ and $\tilde{\kappa} q_n < x_1$. Let κ be a positive constant such that $\kappa < \tilde{\kappa}$, and let $x \in [-\kappa q_n, \kappa q_n]$. We choose δ so that $0 < \delta < \min\{C_1/2, \tilde{\delta}\}$ where $\tilde{\delta}$ and C_1 are the constants in the lemma and in (I), (ii), respectively. Let

$$J = \{k;\, |x - x_k| < \delta q_n/n\},$$

$$J(j) = \{k;\, j\,\delta q_n/n \leqslant |x - x_k| < (j+1)\,\delta q_n/n,\ |x_k| \leqslant \kappa q_n\},$$

$$j = 1, 2, ...,$$

$$I = \{k;\, \delta q_n/n \leqslant |x - x_k|,\ \kappa q_n < |x_k|\}.$$

The sets J, $J(j)$ and I may depend on x and n. The set J contains at most one element and each of the sets $J(j)$, $j=1, 2, \dots$ contains at most two elements, and $\{1, 2, \dots, n\} = \bigcup_{j=0}^{\lambda(n)} J(j) \cup J \cup I$, where $\lambda(n)$ is the smallest number exceeding $2K_1 n/\delta$. Here, K_1 is the constant in (I), (i). Let

$$\sum\nolimits_1 = \sum_{k \in J} R_k(i, r, n; x), \qquad \sum\nolimits_2 = \sum_{j=1}^{\lambda(n)} \sum_{k \in J(j)} R_k(i, r, n; x),$$

$$\sum\nolimits_3 = \sum_{k \in I} R_k(i, r, n; x).$$

Then, $\sum_{k=1}^{n} R_k(i, r, n; x) = \sum_1 + \sum_2 + \sum_3$. To estimate $\sum_p$, $p=1, 2, 3$, we need bounds of the coefficients $e_{ir}(k)$ in (2.2). We shall get the bounds by using the following estimate:

LEMMA 2 [5, Lemma 5]. *Let v be a positive integer, and let $s=0, 1, \dots$. Then,*

$$|\{l_k^v(x)\}^{(s)}|_{x=x_k}| \leqslant CM_n(x_k)^{\langle s \rangle} q_n^{(s-\langle s \rangle)(m-1)}, \qquad k=1, 2, \dots, n.$$

where $\langle s \rangle = 1$ (s: odd), $\langle s \rangle = 0$ (s: even), and $M_n(x_k) = \max\{|x_k|\, q_n^{-2}, |x_k|^{m-1}\}$, and C is independent of n and k.

LEMMA 3. *For $k=1, 2, \dots, n$ and $r=0, 1, \dots, v-1$,*

$$|e_{ir}(v; k, n)| \leqslant C\left(\frac{q_n}{n}\right)^{r-i}, \qquad i=r, r+1, \dots, v-1,$$

where C is independent of n and k.

Proof. We prove this by induction on i. From $h_{rk}^{(r)}(x_k)=1$ and (2.2), it follows that $e_{rr}(k)=1/r!$. Thus, the case $i=r$ holds. By (2.2) and the fact $h_{rk}^{(i)}(x_k)=0$, $r+1 \leqslant i \leqslant v-1$, we easily see

$$e_{ir}(k) = -\sum_{s=r}^{i-1} \frac{1}{(i-s)!} e_{sr}(k)(l_k^v)^{(i-s)}(x_k), \; r+1 \leqslant i \leqslant v-1,$$

Since $M_n(x_k) \leqslant Cq_n^{m-1}$ by (I), (i), it follows from Lemma 2 that $|(l_k^v)^{(s)}(x_k)| \leqslant Cq_n^{s(m-1)} \leqslant C(q_n/n)^{-s}$ for every s, where C is independent of n and k. This inequality and the assumption of induction lead to

$$\begin{aligned}|e_{ir}(k)| &\leqslant C \sum_{s=r}^{i-1} |e_{sr}(k)|\, |(l_k^v)^{(i-s)}(x_k)| \\ &\leqslant C \sum_{s=r}^{i-1} \left(\frac{q_n}{n}\right)^{r-s} \left(\frac{q_n}{n}\right)^{-(i-s)} \leqslant C\left(\frac{q_n}{n}\right)^{r-i},\end{aligned}$$

where C is independent of n and k. Q.E.D.

We continue the proof of Proposition. We first estimate $\sum_1$. We may assume $J \neq \emptyset$. Then, by (I), (i), $J = \{k(x)\}$, where $k(x)$ is the number satisfying $x_{k(x)n} = x_{(x,n)}$. The number $k(x)$ may depend on n. Since $|x - x_{k(x)}| \leq \delta q_n/n$, it follows from Lemma 3 and the mean value theorem that $\sum_1 = R_{k(x)}(i, r, n; x) \leq C\,|p_n'(\xi)/p_n'(x_{k(x)})|^{\nu} \cdot (q_n/n)^r$, where C is independent of n and x, and ξ is between x and $x_{k(x)}$. Since $\kappa < \tilde{\kappa}$ and $x \in [-\kappa q_n, \kappa q_n]$, it follows that $x_{k(x)} \in [-\tilde{\kappa} q_{n-1}, \tilde{\kappa} q_{n-1}]$ for $n \geq n_0$, where n_0 is a number depending only on κ, $\tilde{\kappa}$ and m. By Lemma 1, we have $|p_n'(\xi)/p_n'(x_{k(x)})| \leq C$ for $n \geq n_0$ since $|\xi - x_{k(x)}| \leq |x - x_{k(x)}| \leq \delta q_n/n \leq \tilde{\delta} q_n/n$. Therefore, we have

$$\sum_1 \leq C(q_n/n)^r \tag{2.3}$$

for $x \in [-\kappa q_n, \kappa q_n]$ and $n \geq n_0$, where C is independent of n and x.

We next treat $\sum_2$. Let $1 \leq j \leq \lambda(n)$ and $k \in J(j)$. By Lemma 3, we have $R_k(i, r, n; x) \leq Cj^i\,|p_n(x)/\{(x - x_k)\,p_n'(x_k)\}|^{\nu}\,(q_n/n)^r$ with C independent of n, x and j. We assume $\kappa < \min\{\kappa_1, \kappa_2\}$, where κ_1 and κ_2 are the constants in (I), (ii) and (II), respectively. By (I), (ii), we see that there exists a number n_1 such that if $n \geq n_1$, then $|x - x_{(x,n)}| \leq C_2 q_n/n$ for $x \in [-\kappa q_n, \kappa q_n]$, where n_1 depends only on κ, κ_1 and κ_2. Thus, by (II) we have

$$|p_n(x)| \leq Cw(x)^{-1}\,q_n^{-1/2}, \qquad x \in [-\kappa q_n, \kappa q_n],\, n \geq n_1. \tag{2.4}$$

Since there exists a number n_2 depending only on κ, $\tilde{\kappa}$ and m such that $[-\kappa q_n, \kappa q_n] \subset [-\tilde{\kappa} q_{n-1}, \tilde{\kappa} q_{n-1}]$ for $n \geq n_2$, it follows from Lemma 1 that $|(x - x_k)\,p_n'(x_k)|^{-1} \leq Cj^{-1} q_n^{1/2} w(x_k)$ for $k \in J(j)$ and $n \geq n_2$. Thus, $R_k(i, r, n; x) \leq C\{w(x_k)/w(x)\}^{\nu} j^{i-\nu}(q_n/n)^r \leq Cw(x)^{-\nu} j^{-1} \cdot (q_n/n)^r$ for $x \in [-\kappa q_n, \kappa q_n]$ and $n \geq \max\{n_1, n_2\}$. Since every $J(j)$ has at most two elements, it follows that

$$\sum_2 \leq C\left(\frac{q_n}{n}\right)^r w(x)^{-\nu} \sum_{j=1}^{\lambda(n)} j^{-1} \leq Cw(x)^{-\nu}\left(\frac{q_n}{n}\right)^r \log n \tag{2.5}$$

for $x \in [-\kappa q_n, \kappa q_n]$ and $n \geq \max\{n_1, n_2\}$, where C is independent of n and x.

Lastly, we estimate $\sum_3$. Let $k \in I$ and $x \in [-\kappa q_n, \kappa q_n]$. Since $|x - x_k| \leq (\kappa + K_1)\,q_n$ for every k, it follows from Lemma 3 that $R_k(i, r, n; x) \leq C\,|p_n(x)/\{(x - x_k)\,p_n'(x_k)\}|^{\nu}\,(q_n/n)^{r-i}\,q_n^i$ with C independent of x and n. By (2.4) and $|x - x_k| \geq \delta q_n/n$, we have $R_k(i, r, n; x) \leq Cw(x)^{-\nu} n^{\nu - r + i} q_n^{r - 3\nu/2}\,|p_n'(x_k)|^{-\nu}$ and thus,

$$\sum_3 \leq Cw(x)^{-\nu} n^{\nu - r + i} q_n^{r - 3\nu/2} \sum_{k \in I} |p_n'(x_k)|^{-\nu}$$

for $x \in [-\kappa q_n, \kappa q_n]$, where C is independent of n and x. The sum $\sum_{k \in I} |p_n'(x_k)|^{-\nu}$ is treated by the following lemma.

Let λ_{kn}, $k = 1, 2, \ldots$ be the Cotes numbers which appear in the Gauss–Jacobi quadrature formula

$$\sum_{k=1}^{n} p(x_{kn}) \lambda_{kn} = \int_{-\infty}^{\infty} p(x) w^2(x) \, dx$$

valid for all polynomials $p(x)$ of degree at most $2n-1$ (cf. [11]).

LEMMA 4 [5, Lemma 7]. *Let $\tau > 0$. Then,*

$$\sum_{k:\, |x_k| \geqslant \tau q_n} p_n'(x_k)^{-2} \leqslant C q_n^{-2m+3} w^2(\tau q_n),$$

where C is independent of n.

By the lemma and $\nu/2 \geqslant 1$, we have

$$\sum_{k \in I} |p_n'(x_k)|^{-\nu} \leqslant \left\{ \sum_{k \in I} |p_n'(x_k)|^{-2} \right\}^{\nu/2} n^{1/2}$$
$$\leqslant C\{q_n^{-2m+3} w^2(\kappa q_n)\}^{\nu/2} n^{1/2} = C q_n^{(-2m+3)\nu/2} e^{-\mu n} n^{1/2},$$

where C is independent of n and x, and $\mu = \nu \kappa^m m^{-1}$. Therefore, we have

$$\sum\nolimits_3 \leqslant C w(x)^{-\nu} \left(\frac{q_n}{n}\right)^r e^{-\mu n} n^{i+1/2} \tag{2.6}$$

for $x \in [-\kappa q_n, \kappa q_n]$, where C is independent of n and x. The proof of Proposition is concluded by combining (2.3), (2.5) and (2.6).

We remark that by a more refined estimate on $\sum_2$ we can get $\sum_{k=1}^{n} |h_{rkn}(\nu; x)| \leqslant C\{x^{m-1} w(x)^{-\nu} + \log n\}(q_n/n)^r$ for $x \in [-\kappa q_n, \kappa q_n]$. For the sake of brevity, we omit details.

The fundamental estimate (1.1) established now allows us to prove the Theorem. Let N be a non-negative integer, and let λ and μ be constants such that $0 < \lambda < \mu$. Let $P^*(x)$ be the polynomial of best approximation of order $n-1$ to $f \in C^N(\mathbf{R})$ on the interval $[-\mu q_n, \mu q_n]$. We put $g(t) = f(\mu q_n t)$ and $R^*(t) = P^*(\mu q_n t)$. Then, we note that $R^*(t)$ is the polynomial of best approximation of order $n-1$ to $g(t)$ on the interval $[-1, 1]$. Applying (IV), (i) and changing variable $x = \mu q_n t$, we have

$$
\begin{aligned}
|g^{(j)}(t)-R^{*(j)}(t)| &= (\mu q_n)^j \,|f^{(j)}(x)-P^{*(j)}(x)| \\
&\leqslant C_6(n-1)^{-N}\left\{\Delta_{n-1}\left(\frac{x}{\mu q_n}\right)\right\}^{-j} \\
&\quad\times(\mu q_n)^N E_{n-1-N}(f^{(N)};[-\mu q_n,\mu q_n]) \\
&\leqslant C(n-1)^{-N}\left\{\Delta_{n-1}\left(\frac{x}{\mu q_n}\right)\right\}^{-j} \\
&\quad\times(\mu q_n)^N \omega\left([-\mu q_n,\mu q_n];f^{(N)};\frac{q_n}{n}\right)
\end{aligned}
$$

for $j=0, 1, ..., N$, $n-1>N$ and $|x|\leqslant \mu q_n$, where C is a constant depending only on μ and N. Here, we used Jackson's theorem (cf. [12, 5.1(1)]) and the fact $E_{n-1-N}(g^{(N)};[-1,1])=(\mu q_n)^N\cdot E_{n-1-N}(f^{(N)};[-\mu q_n,\mu q_n])$. For $|x|\leqslant\lambda q_n$, we have $\Delta_{n-1}(x/(\mu q_n))\geqslant(n-1)^{-1}\{1-(\lambda/\mu)^2\}^{1/2}$. Thus, if $0<\lambda<\mu$, then for $|x|\leqslant\lambda q_n$,

$$
|f^{(j)}(x)-P^{*(j)}(x)|\leqslant C\left(\frac{q_n}{n}\right)^{N-j}\omega\left([-\mu q_n,\mu q_n];f^{(N)};\frac{q_n}{n}\right), \tag{2.7}
$$

for $j=0, 1, ..., N$ and $n-1>N$, where C is a constant depending only on N, λ and μ.

Let $\delta(x)=|x|+1$ for $|x|\leqslant 1$ and $\delta(x)=|x|+|x|^{1-m}$ for $|x|>1$. The function $\delta(x)$ has first been introduced by [1] for the case $m=2$. Let $x\in\mathbf{R}$ be fixed. We apply (III) to the polynomial $R(t)=L_n(\delta(x)t)-P^*(\delta(x)t)$ of degree at most $\nu n-1$. Then, we have $|R^{(j)}(t)|\leqslant C_5\,\Delta_{\nu n-1}(t)^{-j}\max_{|s|\leqslant 1}|R(s)|$ for $j=0, 1, ...$ and $|t|\leqslant 1$. We use this inequality for $t=x/\delta(x)$. Then, we easily see that for $j=0, 1, ...,$

$$
\begin{aligned}
&|L_n^{(j)}(x)-P^{*(j)}(x)| \\
&\quad\leqslant C_5\left\{\delta(x)\,\Delta_{\nu n-1}\left(\frac{x}{\delta(x)}\right)\right\}^{-j}\max_{|u|\leqslant\delta(x)}|L_n(u)-P^*(u)| \\
&\quad\leqslant C\frac{n^j}{(\delta(x)^2-x^2)^{j/2}}\max_{|u|\leqslant\delta(x)}|L_n(u)-P^*(u)| \\
&\quad\leqslant C(1+|x|^{j(m-2)/2})\,n^j\max_{|u|\leqslant\delta(x)}|L_n(u)-P^*(u)|,
\end{aligned}
\tag{2.8}
$$

where C is a constant depending only on j and ν. Let K be a constant such that $K_1<K$, and let c be a constant such that $0<c<\min\{K_1,\kappa\}$, where

K_1 and κ are the constants in (I), (i) and Proposition, respectively. By (2.7) and (2.8), we have, for $|x| \leqslant cq_n$ and $j = 0, 1, ..., N$,

$$\begin{aligned}
&|L_n^{(j)}(x) - f^{(j)}(x)| \\
&\leqslant |L_n^{(j)}(x) - P^{*(j)}(x)| + |P^{*(j)}(x) - f^{(j)}(x)| \\
&\leqslant C\left\{(1+|x|^{j(m-2)/2})\, n^j \max_{|u| \leqslant \delta(x)} |L_n(u) - P^*(u)| \right. \\
&\quad \left. + \left(\frac{q_n}{n}\right)^{N-j} \omega\left([-Kq_n, Kq_n]; f^{(N)}; \frac{q_n}{n}\right)\right\}, \qquad (2.9)
\end{aligned}$$

where C is a constant independent of n, x and f.

It is enough to estimate $|L_n(u) - P^*(u)|$ for $|u| \leqslant \delta(x)$. Since the degree of $P^*(u)$ does not exceed $\nu n - 1$, it follows that $L_n(\nu - 1, \nu; P^*, u) = P^*(u)$, which leads to

$$\begin{aligned}
L_n(u) - P^*(u) &= L_n(u) - L_n(\nu - 1, \nu; P^*, u) \\
&= \sum_{k=1}^{n} \sum_{r=0}^{l} \{f^{(r)}(x_k) - P^{*(r)}(x_k)\}\, h_{rk}(u) \\
&\quad - \sum_{k=1}^{n} \sum_{r=l+1}^{\nu-1} P^{*(r)}(x_k)\, h_{rk}(u). \qquad (2.10)
\end{aligned}$$

We note that if $\nu - 1 = l$, then the second sum vanishes. Let N be an integer such that $l \leqslant N$. Since $|x_k| \leqslant K_1 q_n$ for all k by (I), (i), it follows from (2.7) that for $k = 1, 2, ..., n$ and $r = 0, 1, ..., l$,

$$\begin{aligned}
&|f^{(r)}(x_k) - P^{*(r)}(x_k)| \\
&\leqslant C\left(\frac{q_n}{n}\right)^{N-r} \omega\left([-Kq_n, Kq_n]; f^{(N)}; \frac{q_n}{n}\right), \qquad (2.11)
\end{aligned}$$

where C is a positive constant depending only on N, K_1 and K. If $r > l$, then by (IV), (ii) and by changing variables,

$$|P^{*(r)}(x_k)| \leqslant C\left(\frac{q_n}{n}\right)^{l-r} \omega\left([-Kq_n, Kq_n]; f^{(l)}; \frac{q_n}{n}\right), \qquad (2.12)$$

for $k=1, 2, ..., n$, where C is a positive constant depending only on r, K_1 and K. Applying the estimates (2.11) and (2.12) to the expression (2.10), we have

$$|L_n(u)-P^*(u)| \leqslant \begin{cases} C\omega_N\left(\frac{q_n}{n}\right) \sum_{r=0}^{\nu-1} \left(\frac{q_n}{n}\right)^{N-r} \sum_{k=1}^{n} |h_{rk}(u)| & (\nu-1=l), \\ C\omega_l\left(\frac{q_n}{n}\right) \sum_{r=0}^{\nu-1} \left(\frac{q_n}{n}\right)^{l-r} \sum_{k=1}^{n} |h_{rk}(u)| & (\nu-1>l), \end{cases} \tag{2.13}$$

where C is a constant independent of n, x and f, and $\omega_j(q_n/n)$ stands for $\omega([Kq_n, Kq_n]; f^{(j)}; q_n/n)$. Note that there exists a number n_0 such that if $n \geqslant n_0$ then $\delta(x) \leqslant \kappa q_n$ for x with $|x| \leqslant cq_n$. Then, by Proposition and the inequality $\max_{|u| \leqslant \delta(x)} e^{\nu u^m/2} \leqslant Ce^{\nu x^m/2}$ with C independent of x, we completes the proof of Theorem.

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